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Production of Heavy Quarks Close to Threshold*

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Abstract

We calculate production by vector and axial currents of heavy quark pairs ($c\bar{c}$, $b\bar{b}$, $t\bar{t}$) close to threshold. We take into account strong interaction contributions (including radiative corrections and leading nonperturbative effects) by using the Fermi-Watson final state interaction theorem. We use the results obtained to compare with experiment for open production of $c\bar{c}$, $b\bar{b}$ near threshold, and to give a reliable estimate of the so-called "threshold effects" contribution to vector and axial correlators, for $t\bar{t}$, *i.e.*, the contribution of regions close to $4\,m_t^2$ to $\Pi(t)$, for small values of t ($0 < t \lesssim M_Z^2$).

14.40.Gx, 12.38.Bx, 12.38.Lg, 13.20.Gd

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I. INTRODUCTION

In this paper we consider the production of a pair $q\bar{q}$ of heavy quarks close to threshold; that is to say, for small values of $|v|$ where

$$v \equiv \sqrt{1 - 4m^2/s}\ . \tag{1.1}$$

Here m is the quark mass, and $s^{1/2}$ the center of mass energy of the $q\bar{q}$ pair. Above threshold v coincides with the velocity of either quark; but we also study the production of $q\bar{q}$ bound states.

We take the quarks to be heavy, so that we may be able to apply a perturbative QCD analysis to them. Thus we study production of $c\bar{c}$, $b\bar{b}$ and $t\bar{t}$. (For the last case, and in this first paper, we will neglect the effects of t decay). With a view to applications to production by e^+e^- collisions, we take the $q\bar{q}$ to be produced by either a vector V or axial A current[1]:

$$\begin{aligned} V(x) &= \bar{q}(x)\ \gamma_\mu\ q(x) \\ A(x) &= \bar{q}(x)\ \gamma_\mu \gamma_5\ q(x). \end{aligned} \tag{1.2}$$

Sum over omitted color indices is understood. We will then study the correlators,

$$\begin{aligned} \Pi^V_{\mu\nu}(p) &= i \int d^4x\ e^{ip\cdot x} \langle vac|\ T\, V_\mu(x) V_\nu(0)\ |vac\rangle \\ &= (-p^2 g_{\mu\nu} + p_\mu p_\nu)\, \Pi_V(p^2)\ , \end{aligned} \tag{1.3}$$

$$\begin{aligned} \Pi^A_{\mu\nu}(p) &= i \int d^4x\ e^{ip\cdot x} \langle vac|\ T\, A_\mu(x) A_\nu(0)\ |vac\rangle \\ &= (-p^2 g_{\mu\nu} + p_\mu p_\nu)\, \Pi_A(p^2) + p_\mu p_\nu\, \Pi_P(p^2)\ , \end{aligned} \tag{1.4}$$

where $|vac\rangle$ denotes the physical vacuum. As is known, the production cross section for $e^+e^- \to q\bar{q}$ may be straightforwardly written in terms of the imaginary parts of the correlation functions

$$\operatorname{Im} \Pi(p^2)\ , \qquad \Pi = \Pi_{V,A,P}\ \ ; \tag{1.5}$$

[1]Our results may be extended with little effort to scalar or pseudoscalar correlators.

only Π_V, Π_A give sizeable contributions, and they are the quantities we will study here.

To lowest order in perturbation theory (*i.e.*, the parton model) we may neglect the interactions of $q\bar{q}$. The Π are then obtained with a simple one-loop evaluation and one has

$$\mathrm{Im}\,\Pi_V^{(0)}(s) = \frac{N_c}{12\,\pi}\,\frac{v(3-v^2)}{2}\,\theta(s-4\,m^2)\quad, \tag{1.6.a}$$

$$\mathrm{Im}\,\Pi_A^{(0)}(s) = \frac{N_c}{12\,\pi}\,v^3\,\theta(s-4\,m^2)\quad. \tag{1.6.b}$$

Here $N_c = \#\text{ colors } = 3$, and the subscript zero in the Π indicates that the strong interactions are neglected. Of course strong interactions are most important near threshold: its incorporation is precisely the subject of the present paper.

This article is organized as follows. In section 2 we give the expression for the contribution of the $q\bar{q}$ bound states to the $\mathrm{Im}\,\Pi$. This, we hope, will serve to clear some of the misunderstandings found in the standard literature. We also deal, in this section, with the $\mathrm{Im}\,\Pi(s)$ above threshold, but in the nonrelativistic regime ($v^2 \ll 1$), explaining the use of the final state interaction theorem to incorporate strong interactions, which again should clarify some of the existing fog.

In section 3, we use the known evaluations of wave function for bound states, and the one obtained here in the continuum, to give explicit formulas for $\mathrm{Im}\,\Pi(p^2)$ below threshold and above. In this last case radiative and nonperturbative contributions are included for the first time. The article is concluded in section 4, where we discuss in detail the important case of $\mathrm{Im}\,\Pi_V(p^2)$ above threshold and the contribution of this region to the evaluation of $\Pi_V(q^2)$ for $q^2 \ll m^2$. An Appendix is also provided for the technical details of the calculations.

II. THE IMAGINARY PART OF Π AROUND THRESHOLD

A. Bound state contributions to $\mathrm{Im}\,\Pi$.

We will carry over the detailed calculations for $\mathrm{Im}\,\Pi_V$; then we will indicate the corresponding results for $\mathrm{Im}\,\Pi_A$. If we have a bound state of $q\bar{q}$ with momentum k, $k^2 = M^2$, and third component of spin λ, normalized to

$$\langle k, \lambda \mid k', \lambda' \rangle = 2\, k_0\, \delta_{\lambda\lambda'}\, \delta(\vec{k} - \vec{k}') \ ,$$

then its contribution to $\mathrm{Im}\, \Pi^{V}_{\mu\nu}(p)$ is

$$\begin{aligned} \mathrm{Im}\, \Pi^{V;\, pole}_{\mu\nu}(p) &= \frac{1}{2} \int d^4x\; e^{i\, p\cdot x} \sum_{\lambda} \int \frac{d^3k}{2\, k_0}\; \langle vac |\; J_\mu(x)\, | k, \lambda \rangle\; \langle k, \lambda |\; J_\nu(0)\, | vac \rangle \\ &= \frac{1}{2} (2\pi)^4 \delta(p^2 - M^2) \sum_{\lambda}\; V_\mu(p, \lambda)\; V^*_\nu(p, \lambda) \quad . \end{aligned} \tag{2.1}$$

In the above equation,

$$V_\mu(p, \lambda) \equiv \langle vac |\; \bar{q}(0) \gamma_\mu q(0)\; | p, \lambda \rangle \tag{2.2}$$

is the (p-space) wave function of the bound state by *definition*. The connection with the x-space wave function may be carried over immediately in the *nonrelativistic* limit. In the c.m. referencial one thus finds, for *e.g.* $\lambda = +1$, and with $k = p_1 + p_2$,

$$\begin{aligned} &\langle vac |\; \bar{q}(0) \gamma_\mu q(0)\; | k, \lambda = 1 \rangle \\ &\qquad = \frac{\sqrt{N_c}}{(2\, \pi)^{3/2}} \sqrt{\frac{k_0}{2\, p_{10}\, p_{20}}}\; \overline{v}(p_1, 1/2)\; \gamma_\mu\; u(p_2, 1/2)\; \Psi(0) \ . \end{aligned} \tag{2.3}$$

Here $\Psi(0)$ is the x-space wave function evaluated at $\vec{r} = 0$: thus, only states with $\ell = 0$ contribute.

It is perhaps not idle to note that Eqs. (2.1, 2.3) are exact, the last if taken as a definition of $\Psi(0)$. Substituing one into the other, we finally obtain

$$\mathrm{Im}\, \Pi^{pole}_{V}(p^2) = \frac{N_c}{M}\; \delta(p^2 - M^2)\; |R_0(0)|^2 \ , \tag{2.4.a}$$

and $R_0 = (4\, \pi)^{1/2}\, \Psi_{\ell=0}$ is the radial wave function.

For the axial correlator the evaluation is slightly more complicated because the orbital angular momentum of the bound states is $\ell = 1$. One finds,

$$\begin{aligned} &\mathrm{Im}\, \Pi^{pole}_{A}(p^2) = \frac{3\, N_c}{2\, m^2\, M}\; \delta(p^2 - M^2)\; \left| R^{\,\prime}_{1}(0) \right|^2 \ , \\ &R^{\,\prime}_{1}(r) = \partial R_1(r) \, / \, \partial r \quad ; \end{aligned} \tag{2.4.b}$$

R_1 is the radial part of the $\ell = 1$ wave function. In the nonrelativistic limit the R_ℓ are normalized to

$$\int_0^\infty dr\; r^2\, |\; R_\ell(r)\; |^2 = 1 \ .$$

B. $\mathrm{Im}\,\Pi$ above threshold.

We will give a detailed discussion for the case where we have the vector correlator. Moreover, we will consider only the production via a virtual photon, neglecting the contribution of the Z. (This only for ease of discussion; the results will be valid quite generally.) In these circumstances the quantity Im $\Pi^V_{\mu\nu}$ may be considered, for $p^2 > 4\,m^2$, to be proportional to the square of the production amplitude $\gamma^* \to q\bar{q}$:

$$\mathrm{Im}\ \Pi_{\mu\nu} \sim |\ \langle q\bar{q}|\ S\ |\gamma^*\rangle\ |^2 \quad .$$

We will consider that the interactions involved in this process are two: the electromagnetic

$$H_{I\,\mathrm{em}} \;=\; e\,Q_f \int d^{\,3}x\ \bar{q}(x)\,\gamma_\mu\,q(x)\,A^\mu(x) \quad ,$$

and the QCD interaction described by a Hamiltonian H_{IQCD} that will be specified later. The final state interaction theorem then asserts that the amplitude $\langle q\bar{q}|\,S\,|\gamma^*\rangle$ may be evaluated, to first order in $H_{I\,\mathrm{em}}$, but *to all orders* in H_{IQCD}, by means of the expression

$$\langle q\bar{q}|\,S\,|\gamma^*\rangle \;=\; i\langle\Psi|\ H_{I\,\mathrm{em}}\ |\gamma^*\rangle \quad , \tag{2.5}$$

where $|\Psi\rangle$ is an *exact* solution of the Lippmann-Schwinger equation for $q\bar{q}$ states subject to the strong interaction:

$$|\Psi\rangle = |\Psi^{(0)}\rangle + \frac{1}{E - H_0}\ H_{IQCD}\ |\Psi\rangle \quad . \tag{2.6}$$

(In our case, and because only one wave contributes, we need not specify the boundary conditions in Eq. (2.6)).

There is now a complication, as compared to the bound state case: $|\Psi\rangle$ may now contain, besides $q\bar{q}$, $q\bar{q} + n$ gluons. To the order we will be working and in the nonrelativistic approximation , the states $|q\bar{q}+n\ \mathrm{gluons}\rangle$ may however be neglected. The reason is that the amplitude for radiation of a gluon by *heavy* quarks is proportional to the velocities $v_q + v_{\bar{q}}$, so that the contribution of these processes to Im $\Pi_{\mu\nu}$ will be of order $(v_q + v_{\bar{q}})^2 \sim v^2$, *i.e*, of the order of the relativistic corrections, which we are neglecting. Therefore, in the

nonrelativistic limit the state $|\Psi\rangle$ may be considered to consist only of $q\bar{q}$, and can thus be represented by a wave function. We then obtain

$$\mathrm{Im}\,\Pi_V(s) = |\,R_{k0}(0)\,|^2\;\mathrm{Im}\,\Pi_V^{(0)}(s) \quad , \tag{2.7.a}$$

and also

$$\mathrm{Im}\,\Pi_A(s) = |\,3\,R^{\,\prime}_{k1}(0)/m\,v\,|^2\;\mathrm{Im}\,\Pi_A^{(0)}(s) \quad . \tag{2.7.b}$$

Here $R_{k\ell}$ is the radial part of the continuum wave function with $k = m\,v$, and normalized to

$$\int_0^\infty dr\; r^2\, R^*_{k\ell}(r)\, R_{k'\ell}(r) = \frac{2\,\pi}{k^2}\;\delta(k-k') \quad . \tag{2.8}$$

The $\mathrm{Im}\,\Pi^{(0)}$ are as given in Eq. (1.6). Again we would like to emphasize that Eqs. (2.7) are exact (in the nonrelativistic limit); approximations enter when evaluating the $R_{k\ell}$, which will be the subject of the next section.

III. THE $q\bar{q}$ WAVE FUNCTION CLOSE TO THRESHOLD

A. The QCD interaction for heavy quarks at short distances.

It has been known for a long time that the *short distance* interactions of a pair of heavy quarks may be described by perturbation theory; the *leading* nonperturbative corrections are then implemented taking into account the nonzero values of quark and gluon condensates in the physical vacuum:

$$\begin{aligned}
\langle q\bar{q}\rangle \;&\equiv\; \langle vac|\;:\bar{q}(0)\,q(0):\;|vac\rangle \quad ,\\
\langle \alpha_s\,G^2\rangle \;&\equiv\; \alpha_s\langle vac|\;:G_{\mu\nu}(0)\,G^{\mu\nu}(0):\;|vac\rangle \quad .
\end{aligned}$$

The details may be found in the classical SVZ papers[1] where the correlators $\Pi_{\mu\nu}$ are directly studied using the operator product expansion techniques; or the work of Leutwyler[2] and Voloshin[3] where the Green's function method is employed to study the $q\bar{q}$ bound states. In particular, these last authors explicitly prove that no potential may describe the *short*

distance nonperturbative corrections to the $q\bar{q}$ spectrum and wave function; instead, one has to use an effective interaction which in the nonrelativistic limit is given by

$$H_{INP} = -g\,\vec{r}\,\vec{\mathcal{E}}_a\,t^a \quad , \tag{3.1}$$

with $\mathcal{E}$ the chromoelectric field. One then takes,

$$\langle vac|\,\vec{\mathcal{E}}\,|vac\rangle = 0 \quad , \qquad \langle vac|\,\vec{\mathcal{E}}^{\,2}\,|vac\rangle \;\sim\; \langle \alpha_s\,G^2\rangle \; .$$

(In our case the contribution of the quark condensate is negligible). From a practical point of view it has been made apparent in the detailed evaluations of Ref. [4] that a calculation of $q\bar{q}$ states, based on perturbation theory, and supplemented by the leading nonperturbative corrections, as given by Eq. (3.1), yields an excellent, essentially parameter–free description of the bound states of $c\bar{c}$ with $n = 1$ (n being the principal quantum number) and of $b\bar{b}$ states with $n = 1, 2$; $\ell = 0, 1$. As was already known from the work of Refs. [2,3], the analysis breaks down for higher excited states where nonperturbative contributions get out of hand and calculation from first principles becomes impossible. This occurs for $n > 1$ in $c\bar{c}$, and $n > 2$ for $b\bar{b}$. For $t\bar{t}$ the distances involved are so short that a rigorous calculation becomes possible up to $n \sim 5$.

Besides the nonperturbative contributions described by Eq. (3.1) we require also the interaction deduced using perturbation theory. In the nonrelativistic regime, and including one-loop corrections[2], we have the Hamiltonian

$$\begin{aligned} H &= H^{(0)}_{eff} + H_1 \quad , \\ H^{(0)}_{eff} &= -\,\frac{1}{m}\Delta \;-\; \frac{C_F\,\widetilde{\alpha}_s(\mu^2)}{r} \quad , \\ H_1 &= -\,\frac{C_F\beta_0\alpha_s^2}{2\pi}\,\frac{\ln r\mu}{r} \quad ; \end{aligned} \tag{3.2}$$

here $\widetilde{\alpha}_s$ includes part of the radiative corrections,

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[2]The radiative corrections to the $q\bar{q}$ potential have been obtained by a number of authors; cf. Ref. [5] and Ref. [4] where they are completed and summarized.

$$\widetilde{\alpha}_s(\mu^2) = \alpha_s(\mu^2)\ \left[1 + \frac{a_1 + \gamma_E \beta_0/2}{\pi}\ \alpha_s(\mu^2)\right] \quad ;$$
$$\beta_0 = \frac{11\, C_A - 4\, T_F\, n_f}{3} \quad , \quad a_1 = \frac{31\, C_A - 20\, T_F\, n_f}{36} \quad ; \quad C_F = \frac{4}{3},\ C_A = 3,\ T_F = \frac{1}{2} \quad ,$$

and n_f is the number of quark flavors with masses much smaller than m. The reason why we include part of the radiative corrections in $H_{eff}^{(0)}$ is that this Hamiltonian, being Coulombic, may (and will) be solved exactly, whereas H_1 and H_{INP} have to be incorporated in perturbation theory.

A last point about Eq. (3.2) is the meaning of the parameter m there. As follows from the analysis of Ref. [4], this m has to be interpreted as the *pole* mass. That is to say, if $S(\not{p})$ is the quark propagator, in perturbation theory, then m is such that $S^{-1}\,(\not{p} = m) = 0$. This m may be related to the $\overline{\mathrm{MS}}$ mass, $\overline{m}(\overline{m}^2)$ through the formula[6]

$$m = \overline{m}(\overline{m}^2) \left\{ 1 + \frac{C_F \alpha_s(m^2)}{\pi} + (K - 2\, C_F) \left[\frac{\alpha_s(m^2)}{\pi}\right]^2 + \dots \right\} \quad . \tag{3.3}$$

Here $K \sim 13.5$ (an exact formula for K may be found in Ref. [6]). As for the *numerical* values of the masses, the analysis of Ref. [4] gives

$$m_c = 1570 \pm 60\ MeV \quad ; \quad m_b = 4906 \pm 85\ MeV \quad , \tag{3.4}$$

which corresponds to the $\overline{\mathrm{MS}}$ masses

$$\overline{m}_c(\overline{m}_c^2) = 1306 \pm 40\ MeV \quad , \quad \overline{m}_b(\overline{m}_b^2) = 4397 \pm 40\ MeV \quad ,$$

of course compatible with (but more precise than) the values obtained with the SVZ method[1,7]

$$\overline{m}_c(\overline{m}_c^2) = 1270 \pm 50\ MeV \quad , \quad \overline{m}_b(\overline{m}_b^2) = 4250 \pm 100\ MeV \quad .$$

We will thus take Eq. (3.4) as our input. For the top quark we elect to choose

$$m_t = 165 \pm 15\ GeV \quad . \tag{3.5}$$

B. The bound state wave functions.

The energy levels calculated with Eqs. (3.2), (3.1) are[4]

$$E_{n\ell} = 2m \left\{ 1 - \frac{C_F^2\,\widetilde{\alpha}_s(\mu^2)^2}{8\,n^2} - \frac{C_F\beta_0\alpha_s^2\widetilde{\alpha}_s}{8\,\pi\,n^2}\left[\ln\frac{\mu\,n}{mC_F\widetilde{\alpha}_s} + \psi(n+\ell+1)\right] \right.$$
$$\left. + \frac{\pi\,\epsilon_{n\ell}\,n^6\langle\alpha_s G^2\rangle}{2\,(mC_F\widetilde{\alpha}_s)^4} \right\} . \tag{3.6}$$

Here the ϵ are numbers of order unity:

$$\epsilon_{10} = \frac{624}{425}\ ; \quad \epsilon_{20} = \frac{1051}{663}\ ; \quad \epsilon_{30} = \frac{769456}{463239}\ ; \quad \epsilon_{21} = \frac{9929}{9945}\ ; \ldots$$

other $\epsilon_{n\ell}$ may be found in Ref. [4], and an analytic expression in Ref. [2]. For the wave functions, the details are given in Ref. [3] and, especially, in the second paper of Ref. [4]. One has

$$\begin{aligned}
\bar{R}_{n\ell}(r) &= \frac{2}{n^2\,a(n,\ell)^{3/2}}\sqrt{\frac{(n-\ell-1)!}{(n+\ell)!}}\,\rho_{n\ell}^{\ell}\,e^{-\rho_{n\ell}/2}\,L_{n-\ell-1}^{2\ell+1}(\rho_{n\ell})\ , \\
\rho_{n\ell} &= 2\,r/a(n,\ell)\ ; \\
a(n,\ell) &= \frac{2}{mC_F\widetilde{\alpha}_s(\mu^2)}\left\{1 - \frac{\ln(n\mu/mC_F\widetilde{\alpha}_s) + \psi(n+\ell+1) - 1}{2\pi}\,\beta_0\alpha_s\right\} .
\end{aligned} \tag{3.7}$$

This includes the radiative corrections, as obtained using Eq. (3.2). The full wave function at the origin, including nonperturbative contributions (Eq. (3.1)) is

$$R_{n\ell}(0) = \Big(1 + \delta_{NP}(n,\ell)\Big)\ \bar{R}_{n\ell}(0) \tag{3.8}$$

and, for the first $n,\ \ell$,

$$\begin{aligned}
\delta_{NP}(1,0) &= \left\{\frac{2\,968}{425} + \frac{968\,576}{541\,875}\right\}\ \frac{\pi\langle\alpha_s G^2\rangle}{m^4(C_F\widetilde{\alpha}_s)^6}\ , \\
\delta_{NP}(2,0) &= \left\{\frac{3\,828\,736}{1\,989} + \frac{753\,025\,024}{1\,318\,707}\right\}\ \frac{\pi\langle\alpha_s G^2\rangle}{m^4(C_F\widetilde{\alpha}_s)^6}\ , \\
\delta_{NP}(2,1) &= \left\{\frac{3\,299\,840}{1\,989} + \frac{33\,026\,904\,064}{98\,903\,025}\right\}\ \frac{\pi\langle\alpha_s G^2\rangle}{m^4(C_F\widetilde{\alpha}_s)^6}\ .
\end{aligned}$$

Higher $\delta_{NP}(n,\ell)$ may be found in the second paper of Ref. [4].

C. The $q\bar{q}$ wave function in the continuum.

The calculation of the $\bar{R}_{k\ell}$ with the Hamiltonian of Eq. (3.2) is far from trivial. It is described in some detail in the Appendix. We present here the results for the modulus squared, at $r=0$: we have

$$\left|\bar{R}_{k\ell}(0)\right|^2 = [1+2\,c_\ell(k)]\,\left|\widetilde{R}_{kl}(0)\right|^2 \quad . \tag{3.9}$$

Here $\widetilde{R}$ is evaluated with $H^{(0)}_{eff}$ so that

$$\left|\widetilde{R}_{k0}(0)\right|^2 = \frac{\pi\,C_F\widetilde{\alpha}_s\,/v}{1-e^{-\pi C_F\widetilde{\alpha}_s/v}} \quad , \tag{3.10.a}$$

$$\left|3\,\widetilde{R}'_{k1}(0)/m\,v\right|^2 = \left(1+\frac{C_F^2\,\widetilde{\alpha}_s^{\,2}}{4\,v^2}\right)\,\frac{\pi\,C_F\widetilde{\alpha}_s\,/v}{1-e^{-\pi\,C_F\,\widetilde{\alpha}_s\,/v}} \quad , \tag{3.10.b}$$

$\widetilde{\alpha}_s=\widetilde{\alpha}_s(\mu^2)$ given after Eq. (3.2). The functions $c_\ell(k)$ are plotted in Figs. 1, 2 for $\ell=0,1$, and displayed in detail in the Appendix. For $\ell=0,1$, and small velocities,

$$\begin{aligned}
c_0(k) &= \frac{\beta_0\,\alpha_s}{4\,\pi}\left[\ln\frac{\mu\,a}{2}-1-2\,\gamma_E+\frac{(ka)^2}{12}+\frac{(ka)^4}{40}+\dots\right] , \\
c_1(k) &= \frac{\beta_0\,\alpha_s}{2\,\pi}\Bigg[\left(\ln\frac{\mu a}{2}-2\,\gamma_E\right)\left(-\frac{3}{2}+(ka)^2-(ka)^4\right)-2+\frac{49}{24}\,(ka)^2-\frac{167}{80}\,(ka)^4 \\
&\qquad\qquad +\frac{1}{2\,\pi}\left(ka-(ka)^3+(ka)^5\right)+\dots\Bigg] , \\
a &\equiv \frac{2}{m\,C_F\,\widetilde{\alpha}_s} \quad .
\end{aligned} \tag{3.11}$$

A remarkable property of Eq. (3.11) is that, as $k\to 0$ the relevant scale for $\alpha_s(\mu^2)$ is *not* $\mu\sim k$ (as one would naively guess, and as assumed for instance in Refs. [8,9]), but $\mu\sim 2/a = m\,C_F\widetilde{\alpha}_s$: the interaction *saturates*.

The full wave function is obtained by adding the (leading) nonperturbative contributions. These may be deduced from the the SVZ calculations[1]. One finds

$$|R_{k\ell}(0)|^2 = \left(1+2\,\delta^{NP}_{k\ell}\right)\,\left|\bar{R}_{k\ell}(0)\right|^2 \quad . \tag{3.12}$$

For $\ell=0$ one has

$$\delta_{k0}^{NP} = -\frac{\pi \, \langle \alpha_s \, G^2 \rangle}{192 \, m^4 \, v^6} \quad . \tag{3.13}$$

Just like for the bound states, the nonperturbative correction blows up at threshold ($v \to 0$). Clearly, the calculation ceases to be valid when $\left|\delta^{NP}\right|$ is of the order of magnitude of $|c_\ell|$, *i.e.*, for a critical velocity v_{crit} such that

$$v_{crit} \sim \left(\frac{\pi^2 \langle \alpha_s \, G^2 \rangle}{192 \, \beta_0 \, m^4} \right)^{1/6} \quad . \tag{3.14}$$

IV. NUMERICAL RESULTS

A. Production of $q\bar{q}$ above threshold.

We now consider the quantities

$$\begin{aligned} R_q^V(s) \;&\equiv\; 12 \, \pi \, Q_q^2 \; \mathrm{Im} \, \Pi_V(s) \quad , \\ R_q^A(s) \;&\equiv\; 12 \, \pi \, Q_q^2 \; \mathrm{Im} \, \Pi_A(s) \quad . \end{aligned} \tag{4.1.a}$$

R_q^V is essentially the ratio $(e^+e^- \to \gamma^* \to q\bar{q})/(e^+e^- \to \gamma^* \to \mu^+\mu^-)$. To show clearly the various contributions, we will plot the zeroth and first order (in α_s) expressions for the $R_q(s)$, cf. Eq. (1.6):

$$\begin{aligned} R_q^{V\,(0)}(s) &= N_c \, Q_q^2 \, \frac{v(3 - v^2)}{2} \quad , \\ R_q^{V\,(1)}(s) &\underset{v \to 0}{=} N_c \, Q_q^2 \left\{ \frac{3\,\pi}{4} - \frac{6\,v}{\pi} + \frac{\pi\, v^2}{2} + \ldots \right\} C_F \, \alpha_s(\mu^2) \; , \\ R_q^{V\,(1)}(s) &\underset{v \to 1}{=} N_c \, Q_q^2 \left\{ \frac{3}{4} + \frac{9}{2}(1 - v) + \left(\frac{9}{2} \ln \frac{2}{1 - v} - \frac{3}{8} \right) (1 - v)^2 + \ldots \right\} \frac{C_F \, \alpha_s(\mu^2)}{\pi} \; ; \end{aligned} \tag{4.2.a}$$

$$\begin{aligned} R_q^{A\,(0)}(s) &= N_c \, Q_q^2 \, v^3 \quad , \\ R_q^{A\,(1)}(s) &= N_c \, Q_q^2 \left\{ \pi \, C_F \, \alpha_s(\mu^2) \, v^2 + \ldots \right\} \; . \end{aligned} \tag{4.2.b}$$

The expression for $R_q^{V\,(1)}$ is actually known exactly[10], but Eq. (4.2.a) is accurate enough for our purposes. Only the leading term (in v) in $R_q^{A\,(1)}$ is known, and it is reported in

Eq. (4.2.b). It is worth noting that the Shwinger interpolation[10] to $R_q^{V\,(1)}$, used by some authors, is *not* accurate enough for our purposes. In fact, it reproduces correctly the values of $R_q^{V\,(1)}(s)$ for $v=0,\ \ v=1$, but not the *derivatives*, *i.e.*, the terms in $v,\ v^2;\ (1-v),\ (1-v)^2$. For this reason we take the full (4.2.a) (see below, e.g. in (4.3.a)).

Together with these perturbative evaluations we also plot the "exact" expressions obtained from Eqs. (3.9–3.13):

$$R_q^V(s) \underset{v\to 0}{=} N_c\, Q_q^2 \left\{ \frac{v(3-v^2)}{2} + \left(-\frac{6\,v}{\pi} + \frac{3\,\pi\, v^2}{4} \right) C_F\, \widetilde{\alpha}_s \right\} \left(1 - \frac{\pi \langle \alpha_s\, G^2 \rangle}{192\, m^4\, v^6} \right)$$
$$\times [1 + 2\, c_0(k)]\, \frac{\pi\, C_F \widetilde{\alpha}_s / v}{1 - e^{-\pi C_F \widetilde{\alpha}_s/v}} \qquad , \qquad \widetilde{\alpha}_s = \widetilde{\alpha}_s(\mu^2)\ ; \tag{4.3.a}$$

we have included the known $\mathcal{O}(\alpha_s v,\ \alpha_s v^2)$ corrections, cf. Eq. (4.2.a). For R_q^A,

$$R_q^A(s) \underset{v\to 0}{=} N_c\, Q_q^2\, v^3\, (1 + 2\, \delta_{k1}^{NP})\, [1 + 2\, c_1(k)] \left(1 + \frac{C_F^2\, \widetilde{\alpha}_s^2}{4\, v^2} \right) \frac{\pi\, C_F \widetilde{\alpha}_s / v}{1 - e^{-\pi C_F \widetilde{\alpha}_s/v}}\ . \tag{4.3.b}$$

The quantities δ_{k1}^{NP} may be found in Ref. [11]. The calculation of R_q^V will be made for $q=c, b$; R_q^A will also be evaluated for $q=t$.

Before presenting the results, a few words have to be said about the parameters entering Eqs. (4.2, 4.3). For the masses we will take the values given in Eqs. (3.4, 3.5). For $\alpha_s(\mu^2)$, the "natural" scale is $\mu \sim m\, C_F\, \widetilde{\alpha}_s$, as shown in Eqs. (3.11). Nevertheless, for $c,\ b$ we will choose $\mu = 2\, m_q$ (but see subsection B for a discussion of this). For $t\bar{t}$ production, and since the choice of μ is less relevant now, we will take $\mu = M_Z$; then we may use directly the value of α_s deduced from the Z decays. Thus, we take

$$\alpha_s(M_Z) = 0.119 \pm 0.003\ , \tag{4.4.a}$$

for $t\bar{t}$; and for $b\bar{b}$ and $c\bar{c}$,

$$\alpha_s(\mu^2) = \frac{4\,\pi}{\beta_0\ \ln\, \mu^2/\Lambda^2} \left\{ 1 - \frac{\beta_1\ \ln\ln\, \mu^2/\Lambda^2}{\beta_0^2\ \ln\, \mu^2/\Lambda^2} \right\}\ , \tag{4.4.b}$$

$$\mu = 2\, m_c,\ 2\, m_b \quad ; \quad \Lambda = 200^{+80}_{-60}\ MeV \quad , \quad \beta_1 = 102 - 38\, n_f/3\ .$$

Finally, for $\langle \alpha_s G^2 \rangle$ (which does not play a very important role in our evaluations) we choose the standard value[12]

$$\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020 \ GeV^4 \ . \tag{4.5}$$

As stated several times, our calculations are valid in the nonrelativistic regime, *i.e.*, to corrections $\mathcal{O}(v^2)$. For the numerical evaluations we will take $v < v_{\rm Max}$, $v_{\rm Max} = 1/2$.

B. Comparison with experiment: $c\bar{c}$ and $b\bar{b}$.

The prediction of our calculation (4.3.a) for $c\bar{c}$ is shown in Fig. 3. In order to display the dependence of the calculation on the choice of the renormalization scale, μ, we present these R_c^V for two choices of μ: electing μ such that the radiative corrections vanish, *i.e.*, such that $c_0 = 0$, or fixing $\mu = 2\,m_c$. There is little difference between both choices; here we favour this last choice because it ties with the election at higher energies ($s^{1/2} \gg m_c$) where one takes $\mu = s^{1/2}$. Also shown are the results of a parton model ($R_c^{V\,(0)}$) and parton model plus order $\alpha_s(m_c)$ correction (the last denoted by $R_c^{V\,(1)}$). It is seen that there is a partial cancellation of the Fermi factor,

$$\frac{\pi\, C_F \widetilde{\alpha}_s / v}{1 - e^{-\pi C_F \widetilde{\alpha}_s / v}} \quad ,$$

and the radiative correction, $2\,c_0(k)$ (c.f. Eq. (4.3.a)) in such a way that R_q^V does not differ much from $R_c^{V\,(0)+(1)}$. The nonperturbative contribution is only important right at threshold.

Comparison with experiment is shown in Fig. 4. Although the quality of this is not enchanting as a fit, a few things must be said in its favour. First, the theoretical curve runs, more or less, through the middle of the experimental points. It is clear that the full $R_c^{V\,({\rm exact})}$ is more centered than the purely partonic $R_c^{V\,(0)}$ or what we would have obtained neglecting the radiative correction, $R_c^{V\,(\text{no correc.})}$ (in this last case, the improvement is slight). *On the average*, $R_c^{V\,({\rm exact})}$ represents a good mean of experiment, which is an interesting fact, particularly since part of the dispersion of the experimental points is doubtlessly due to error fluctuations.

Fig. 5 shows the comparison with experiment for $b\bar{b}$. The conclusions are similar to those for $c\bar{c}$: $R_b^{V\,(\mathrm{exact})}$ improves $R_b^{V\,(0)}$ and represents a good average of the experimental points, which are now scantier.

C. $t\bar{t}$ production.

We plot in Figs. 6, 7 the quantities R^V, R^A relevant to production of $t\bar{t}$ by a vector, axial current respectively. It is not possible to use them directly to predict experimental output because our formulas do not take into account the width of t. This can be done with the standard methods[13], and we will present the details separately.

V. THRESHOLD EFFECTS ON "LOW ENERGY" CORRELATORS.

Consider a correlator, $\Pi(t)$. It is possible to prove quite generally that it verifies the relation (*dispersion relation*)

$$\Pi(t) - \Pi(0) = \frac{t}{\pi} \int^{\infty} ds\, \frac{\mathrm{Im}\,\Pi(s)}{s(s-t)} \quad . \tag{5.1}$$

By expanding in a power series in α_s it follows that Eq. (5.1) is also verified order by order in perturbation theory:

$$\Pi^{(n)}(t) - \Pi^{(n)}(0) = \frac{t}{\pi} \int^{\infty} ds\, \frac{\mathrm{Im}\,\Pi^{(n)}(s)}{s(s-t)} \quad . \tag{5.2}$$

When used in this last form, the dispersion relation is little more than a calculational device which may simplify the evaluation of $\Pi^{(n)}$ as, generally speaking, $\mathrm{Im}\,\Pi^{(n)}$ is easier to calculate than $\mathrm{Re}\,\Pi^{(n)}$. This method is followed e.g. in Refs. [10] for $\Pi_V^{(1)}$. When used in the form (5.1), however, a dispersion relation may yield new knowledge. This happens when experimental data may be used as input for $\mathrm{Im}\,\Pi$, thus obtaining values of Π in regions inaccessible to theory, as is the case for evaluations of the hadronic corrections to the anomalous magnetic moment of the muon. Another situation, which is the one we will

encounter here, is when nonperturbative methods are employed to evaluate $\operatorname{Im}\Pi$ (or parts thereof).

Let us define the *threshold effect* contributions to $\Pi(t)-\Pi(0)$, to be denoted by $\Delta_{\rm th}^{(n)}(t)$, to be the quantity

$$\begin{aligned}\Delta_{\rm th}^{(n)}(t) &= \frac{t}{\pi}\int^{s_{\rm M}} ds\, \frac{f^{(n)}(s)}{s(s-t)} \\ f^{(n)}(s) &\equiv \operatorname{Im}\Pi(s) - \sum_{n'=0}^{n} \operatorname{Im}\Pi^{(n')}(s)\end{aligned} \tag{5.3}$$

That is to say, $\Delta_{\rm th}^{(n)}$ incorporates the exact contribution to $\Pi(t)-\Pi(0)$ from a region around threshold, up to an energy $s_{\rm M}$, but subtracting the first n terms in perturbation theory. Clearly, one has

$$\Pi(t)-\Pi(0) = \sum_{n'=0}^{n}\left\{\Pi^{(n')}(t)-\Pi^{(n')}(0)\right\} + \Delta_{\rm th}^{(n)}(t) + \Delta_{\rm h.e}^{(n)} \quad , \tag{5.4}$$

where $\Delta_{\rm h.e}^{(n)}$ would be

$$\Delta_{\rm h.e}^{(n)}(t) = \frac{t}{\pi}\int_{s_{\rm M}}^{\infty} ds\, \frac{\operatorname{Im}\Pi(s) - \sum_{n'=0}^{n}\operatorname{Im}\Pi^{(n')}(s)}{s(s-t)} \quad . \tag{5.5}$$

Expression (5.4) is then useful when we can argue that $\Delta_{\rm h.e}^{(n)}$ is small with respect to $\Delta_{\rm th}^{(n)}$, or (as happens for Π_V) when the difference

$$\operatorname{Im}\Pi(s) - \sum_{n'=0}^{n}\operatorname{Im}\Pi^{(n')}(s)$$

may be calculated for $s > s_{\rm M}$.

It should be clear that $\Delta_{\rm th}^{(n)} + \Delta_{\rm h.e}^{(n)}$ depends on $s_{\rm M}$; only if *exact* evaluations were used that an exact match (and thus cancellation) of the dependence of each of $\Delta_{\rm th}^{(n)}$, $\Delta_{\rm h.e}^{(n)}$ on $s_{\rm M}$ would occur. In favorable cases we expect, however, that the residual dependence would be slight.

A. The axial correlator.

We do not have any information on $\Delta_{\rm h.e}^{(n)}$ for the axial correlator, but we will give results on $\Delta_{{\rm th}\,A}^{(2)}$ for completeness. Separating the contribution from the bound states and the piece above threshold, we write

$$\Delta_{{\rm th}\,A}^{(n)} = \Delta_{{\rm pole}\,A}^{(n)} + \Delta_{{\rm a.t.}\,A}^{(n)} \quad . \tag{5.6}$$

Using Eq. (2.4.b) for $\Delta_{{\rm pole}\,A}^{(2)}$, and Eqs. (4.1), (4.2.b), (4.3.b) for $\Delta_{{\rm a.t.}\,A}^{(2)}$, we obtain

$$\Delta_{{\rm pole}\,A}^{(1)}(t) = \frac{3\,N_c\,t}{2\,m^2\,\pi} \sum_N \frac{1}{M_{\rm N1}^3}\,\frac{1}{M_{\rm N1}^2 - t}\,\left|R'_{\rm N1}(0)\right|^2 \quad , \tag{5.7}$$

where the sum runs over the $\ell = 1$ bound states, and

$$\begin{aligned} \Delta_{{\rm a.t.}\,A}^{(1)}(t) = {}& \frac{N_c\,t}{12\,\pi^2} \int_{s_{\rm th}}^{s_{\rm M}} ds\, \frac{1}{s\,(s-t)} \\ & \times\left\{ v^3 \left(1 - 2\,\delta_{k1}^{NP}\right) [1 + 2\,c_1(k)] \left(1 - \frac{C_F^2\,\widetilde{\alpha}_s^2}{4\,v^2}\right) \frac{\pi C_F \widetilde{\alpha}_s/v}{1 - e^{-\pi C_F \widetilde{\alpha}_s/v}} \right. \\ & \left. -v^3 - \pi C_F \alpha_s v^2 \right\} \quad , \qquad k \equiv mv \quad , \quad v \equiv \sqrt{1 - 4\,m^2/s} \; . \end{aligned} \tag{5.8}$$

For the case of $t\bar{t}$ production, the nonperturbative corrections δ_{k1}^{NP} are negligible, so in this particular case we may approximate the threshold $s_{\rm th}$ by $4\,m_t^2$. Moreover, the spectrum of bound states is Coulombic with a very good approximation up to $N \sim 4$; since bound states with $N > 4$ contribute very little to Eq. (5.7) anyway, we may take the spectrum to be Coulombic all the way. Thus, we may rewrite Eqs. (5.7), (5.8) as

$$\Delta_{{\rm pole}\,A}^{(1)}(t) = \frac{N_c\,t}{m^2\,\pi} \sum_{N=2}^{\infty} \frac{1}{M_{\rm N1}^3\,(M_{\rm N1}^2 - t)}\,\frac{N+1}{N^3 (N-1)\,a(N,1)^5} \quad , \tag{5.9.a}$$

$a(N,1)$ given in Eq. (3.7), and the $M_{\rm N}$ are as in Eq. (3.6) (neglecting the NP piece),

$$M_{\rm N1} = 2\,m \left\{ 1 - \frac{C_F^2 \widetilde{\alpha}_s(\mu^2)^2}{8\,N^2} - \frac{C_F \beta_0 \alpha_s^2 \widetilde{\alpha}_s}{8\,\pi\,N^2} \left[\ln \frac{\mu N}{m C_F \widetilde{\alpha}_s} + \psi(N+2) \right] \right\} \; . \tag{5.9.b}$$

Likewise,

$$\begin{aligned} \Delta_{{\rm a.t.}\,A}^{(1)}(t) = {}& \frac{N_c\,t}{12\,\pi^2} \int_{4\,m^2}^{s_{\rm M}} ds\, \frac{1}{s\,(s-t)} \\ & \times\left\{ v^3 [1 + 2\,c_1(k)] \left(1 - \frac{C_F^2\,\widetilde{\alpha}_s^2}{4\,v^2}\right) \frac{\pi C_F \widetilde{\alpha}_s/v}{1 - e^{-\pi C_F \widetilde{\alpha}_s/v}} - v^3 - \pi C_F \alpha_s v^2 \right\} \; . \end{aligned} \tag{5.10}$$

The practical interest of this is fairly limited so long as we have no reliable estimate of $\Delta^{(1)}_{\mathrm{h.e}\,A}$. We then turn to the vector correlator, for which such an estimate exists.

B. The vector correlator.

The calculation is very much like for the axial case, except that we use Eq. (2.4.a) for the poles, and Eqs. (4.1), (4.2.a), (4.3.a) above threshold. We find, for $t\bar{t}$ and neglecting the gluon condensate contribution,

$$\Delta^{(1)}_{\mathrm{pole}\,V}(t) = \frac{4\,N_c\,t}{\pi} \sum_{N=1}^{\infty} \frac{1}{M^3_{\mathrm{N0}}\,(M^2_{\mathrm{N0}} - t)}\,\frac{1}{N^3\,a(N,0)^3} \quad , \tag{5.11}$$

and now

$$M_{\mathrm{N0}} = 2\,m\left\{1 - \frac{C_F^2\widetilde{\alpha}_s(\mu^2)^2}{8\,N^2} - \frac{C_F\beta_0\alpha_s^2\widetilde{\alpha}_s}{8\,\pi\,N^2}\left[\ln\frac{\mu}{mC_F\widetilde{\alpha}_s} + \psi(N+1)\right]\right\} \;. \tag{5.12}$$

We split the region above thresholds into two parts, a low energy ($l.e.$) and a high energy ($h.e.$) part, according to $v < 1/2$ or $v > 1/2$. Note that $v = 1/2$ occurs for $s = 16\,m^2/3$. Later we will discuss the joining of the two regions.

At low energy we have,

$$\Delta^{(1)}_{\mathrm{l.e.}\,V}(t) = \frac{N_c\,t}{12\,\pi^2}\int_{4\,m^2}^{16\,m^2/3} ds\,\frac{f^{(1)}_{\mathrm{l.e.}}(s)}{s\,(s-t)} \quad , \tag{5.13.a}$$

and, from Eqs. (4.3.a), (4.2.a)

$$\begin{aligned} f^{(1)}_{\mathrm{l.e.}}(s) \equiv{}& R^V_t - R_t^{V\,(0)+(1)} \\ ={}& \frac{\pi C_F\widetilde{\alpha}_s(\mu^2)/v}{1 - e^{-\pi C_F\widetilde{\alpha}_s(\mu^2)/v}}\left\{\frac{v(3-v^2)}{2} + \left(-\frac{6\,v}{\pi} + \frac{3\,\pi v^2}{4}\right)C_F\widetilde{\alpha}_s(\mu)\right\}\,[1 + 2\,c_0(k)] \\ & - \frac{v(3-v^2)}{2} - \left(\frac{3\,\pi}{4} - \frac{6\,v}{\pi} + \frac{\pi v^2}{2}\right)C_F\alpha_s(\mu^2) \quad , \\ & k = mv \quad , \quad v = \sqrt{1 - 4\,m^2/s} \quad , \quad \mu = M_Z \;. \end{aligned} \tag{5.13.b}$$

For high energy there exist evaluations[14] correct to errors $(1-v)^2\ \alpha_s^2$, α_s^4. Subtracting from this the order α_s piece, we may write the result as

$$f^{(1)}_{\mathrm{h.e.}} \equiv \bar{f}_{\mathrm{h.e.}} - f^{(0+1)}_{\mathrm{h.e.}} \quad , \tag{5.14.a}$$

where $f^{(0+1)}_{\rm h.e.}$ is the piece of order $(0+1)$ in α_s,

$$f^{(0+1)}_{\rm h.e.} = -\frac{3}{2}(1-v)^2 + \frac{1}{2}(1-v)^3 + \left[\frac{9}{2}(1-v) + \frac{9}{2}\left(\ln\frac{2}{1-v} - \frac{3}{8}\right)(1-v)^2\right]\frac{C_F\alpha_s}{\pi}\ , \tag{5.14.b}$$

and[14],

$$\bar{f}_{\rm h.e.} = -\frac{3}{2}(1-\overline{v})^2 + \frac{1}{2}(1-\overline{v})^3 + \left[\frac{9}{2}(1-\overline{v}) + \frac{9}{2}\left(\ln\frac{2}{1-\overline{v}} - \frac{3}{8}\right)(1-\overline{v})^2\right]\frac{C_F\alpha_s}{\pi} + r_2\left(\frac{\alpha_s(s)}{\pi}\right)^2 + \tilde{r}_3\left(\frac{\alpha_s}{\pi}\right)^3 + \frac{9}{2}(1-v)C_F\left[8.7\left(\frac{\alpha_s(s)}{\pi}\right)^2 + 45.3\left(\frac{\alpha_s(s)}{\pi}\right)^3\right]\ . \tag{5.14.c}$$

Here,

$$\overline{v} = \sqrt{1-\overline{m}^2(s)/s}\quad ,\quad r_2 = 1.986 - 0.115\, n_f\quad ,$$
$$\tilde{r}_3 = -6.637 - 1.2\, n_f - 0.005\, n_f^2 - 1.24\left(\sum_f Q_f\right)^2\quad ,$$

and the running mass $\overline{m}(\mu)$ is given in terms of m by

$$\overline{m} = m\left[\frac{\alpha_s(\mu)}{\alpha_s(m)}\right]^{d_m}\left\{1 + \frac{A\,\alpha_s(\mu) - (C_F - A)\alpha_s(m)}{\pi}\right\}\quad ,$$
$$A = \frac{\beta_1\gamma_0 - \beta_0\gamma_1}{\beta_0^2}\quad ,\quad d_m = -\frac{\gamma_0}{\beta_0}\quad ,$$
$$\gamma_0 = -3\,C_F\quad ,\quad \gamma_1 = -\frac{3\,C_F^2}{2} - \frac{97\,C_F C_A}{6} + \frac{5\,C_F n_f}{3}$$

Finally,

$$\Delta^{(1)}_{{\rm h.e.}\ V}(t) = \frac{N_c\, t}{12\,\pi^2}\int_{16\,m^2/3}^{\infty} ds\, \frac{f^{(1)}_{\rm h.e.}(s)}{s\,(s-t)}\quad . \tag{5.15}$$

$\Delta^{(1)}_{{\rm pole}\ V}$, $\Delta^{(1)}_{{\rm l.e.}\ V}$, and $\Delta^{(1)}_{{\rm h.e.}\ V}$ should be compared with the $(0+1)$–order direct calculation of Π: for small t,

$$\Pi_1(t) \equiv \sum_{n=0}^{1}\left\{\Pi^{(n)}(t) - \Pi^{(n)}(0)\right\} = \frac{N_c\, t}{12\,\pi^2}\,\frac{1}{5\,m^2}\left\{1 - \frac{3}{28}\,\frac{t}{m^2} + \frac{205}{54}\,\frac{C_F\alpha_s(\mu^2)}{\pi} + \dots\right\} \tag{5.16}$$

Numerically,

$$\frac{3\,M_Z^2}{28\,m^2} = 3.2 \times 10^{-2} \qquad ; \qquad \frac{205}{54}\,\frac{C_F\,\alpha_s(M_Z^2)}{\pi} = 0.19 \pm 0.005 \quad . \tag{5.17}$$

The error in the above equation is due to the experimental error in $\alpha_s(M_Z)$.

As for the $\Delta^{(1)}$, we write, for the various contributions,

$$\Delta^{(1)}(t) = \frac{N_c\,t}{12\,\pi^2}\,\frac{1}{5\,m^2}\,\widehat{\Delta}^{(1)}(t) \quad . \tag{5.18}$$

In this way we may compare directly with Eq. (5.17). Then,

$$\begin{aligned}
\widehat{\Delta}^{(1)}_{\text{pole }V}(0) &= 2.33 \times 10^{-2} \\
\widehat{\Delta}^{(1)}_{\text{l.e. }V}(0) &= 1.56 \times 10^{-2} \\
\widehat{\Delta}^{(1)}_{\text{h.e. }V}(0) &= 2.15 \times 10^{-2}
\end{aligned} \tag{5.19}$$

The dependence of the $\widehat{\Delta}$ on t is very slight, up to $t = M_Z^2$, where we have

$$\begin{aligned}
\widehat{\Delta}^{(1)}_{\text{pole }V}(M_Z^2) &= 2.52 \times 10^{-2} \\
\widehat{\Delta}^{(1)}_{\text{l.e. }V}(M_Z^2) &= 1.67 \times 10^{-2} \\
\widehat{\Delta}^{(1)}_{\text{h.e. }V}(M_Z^2) &= 2.24 \times 10^{-2}
\end{aligned} \tag{5.20}$$

The largest error of the Δ is due to the error in the mass of the t quark, about which little can be done at present. Another source of error is due to extrapolations: we have used Eq. (5.13) for $f_{\text{l.e.}}$ up to $v = 1/2$, and Eq. (5.14) for $f_{\text{h.e.}}$ down to same value of v. We can smooth this rather crude joining of l.e. and h.e. regions by defining

$$f_1 \equiv (1 - v)\,f_{\text{l.e.}} + v\,f_{\text{h.e.}} \tag{5.21.a}$$

and integrating this f_1 over all the interval, from $4\,m^2$ to infinity. Eqs. (5.19) are not changed substantially; we obtain now, with self-explanatory notation,

$$\widehat{\Delta}^{(1)}_{\text{l.e.+h.e. }V\,,\,1}(0) = 4.1 \times 10^{-2} \quad . \tag{5.21.b}$$

Another possibility is to write

$$f_3 \equiv (1 - v^3)\,f_{\text{l.e.}} + v^3\,f_{\text{h.e.}} \quad , \tag{5.22.a}$$

which respects the terms in $v^0,\ v,\ v^2$ which are known at low energy. Then,

$$\widehat{\Delta}^{(1)}_{\text{l.e.+h.e.}\ V\ ,3}(0) = 3.3 \times 10^{-2} \quad . \tag{5.22.b}$$

We consider Eqs. (5.21) to be the more reasonable estimate, and take its difference with Eqs. (5.19), (5.22) to be a measure of the systematic theoretical errors in our calculation[3].

Thus we finally get

$$\begin{aligned}\widehat{\Delta}^{(1)}_{\text{all}\ V}(0) &= \widehat{\Delta}^{(1)}_{\text{pole}\ V}(0) + \widehat{\Delta}^{(1)}_{\text{l.e.}\ V}(0) + \widehat{\Delta}^{(1)}_{\text{h.e.}\ V}(0) \\ &= (\ 4.1 \pm 0.8\ ) \times 10^{-2} \quad .\end{aligned} \tag{5.23}$$

This may be compared with the evaluations of Refs. [8,9,15], which are clearly improved by our results. In particular, the unstabilities noted by Gonzalez-Garcia et al.[15] disappear almost completely. This is due to our use of information from the *high energy* region (which eliminates the uncertainties due to the dependence on an energy cut-off), and inclusion of the radiative corrections, which reduce drastically the arbitrariness of the choice of the scale μ of $\alpha_s(\mu)$.

Although the evaluation is reasonably reliable, it should be clear that the effect is small in the sense that $\widehat{\Delta}_{\text{all}\ V}$ as given by Eq. (5.23) is smaller than the perturbatively known piece, Eq. (5.17), by a factor of about 5.

VI. ACKNOWLEDGEMENTS

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APPENDIX A: WAVE FUNCTIONS IN THE CONTINUUM

We present here some of the technicalities used to solve the radial wave equation in the continuum. Let us recall that we are only interested in obtaining the value of the wave

[3]To which one should add errors due to experimental errors in m_t, $\alpha_s(M_Z)$.

function at the origin for $\ell = 0$, and its derivative at the origin for $\ell = 1$. This simplifies considerably the calculation. The main results of this Appendix were given in Eq. (3.11).

To find the radial solutions $\bar{R}_{k\ell}(r)$ of the Hamiltonian H, as given in Eq. (3.2), we treat H_1 to first order, and use the method of variation of constants. In other words, we want to solve

$$\bar{R}_{k\ell}^{''}(r) + \frac{2}{r}\,\bar{R}_{k\ell}^{'}(r) + \left(-\frac{\ell(\ell+1)}{r^2} + k^2 + \frac{2}{ar} + \frac{2\,\lambda\,\ln r\mu}{a\,r}\right)\,\bar{R}_{k\ell}(r) = 0 \quad ,$$
$$k = mv \ , \quad a = \frac{2}{mC_F\,\tilde{\alpha}_s(\mu^2)} \ , \quad \lambda = \frac{m\,a\,C_F\beta_0\alpha_s^2}{4\pi} \ , \tag{A.1}$$

taking λ infinitesimal. For $\lambda = 0$, the *normalized solution* (see Eq. (2.8)) which is regular at $r = 0$ is

$$\tilde{R}_{k\ell}(r) = e^{i\delta}\,\frac{|\,\Gamma(A_\ell)\,|}{\Gamma(2\ell+2)}\,e^{\pi/2ka}\,e^{-ikr}\,(2kr)^\ell\,M(A_\ell, 2\ell+2, 2ikr) \quad , \tag{A.2}$$

where

$$A_\ell \equiv \ell + 1 + \frac{i}{ka} \quad , \tag{A.3}$$

M is the Kummer function, and δ an arbitrary phase, which we choose to be zero.

Let us now discuss the case $\ell = 0$. The general solution of Eq. (A.1) is

$$\begin{aligned}
&\bar{R}_{k0} = \tilde{R}_{k0} + \delta R_{k0} \quad ,\\
&\delta R_{k0} = e^{\pi/2ka}\,|\,\Gamma(A_0)\,|\,e^{-ikr}\,Y(r) \quad ,\\
&Y(r) = c_0(k)\ M(A_0, 2, 2ikr) + 2ikr\Big[M(A_0, 2, 2ikr)\int_0^r d\rho\,\varphi_1\\
&\qquad\qquad + U(A_0, 2, 2ikr)\int_0^r d\rho\,\varphi_2\Big] \ ,
\end{aligned}$$

where

$$\varphi_1 = UX/W \quad , \quad \varphi_2 = -MX/W \quad , \quad X = \frac{i\lambda}{ka}\,\frac{\ln r\mu}{2ikr}\,M(A_0, 2, 2ikr) \quad ,$$

U is the Kummer function which is singular at $r = 0$, and W is the Wronskian of U and M,

$$W^{-1} = \Gamma(A_0)\,e^{-2ikr}\,(2ikr)^2 \quad .$$

A term in Y of the form $C(k)\,U$ is excluded by the condition of regularity at $r = 0$. The constant c_0 must be determined from the condition of normalization of $\overline{R}_{k0}$, which is equivalent here to the orthogonality of $\widetilde{R}_{k0}$ and $\delta\, R_{k0}$:

$$\int_0^\infty dr\; r^2\, \widetilde{R}^*_{k0}\; \delta R_{k0} \;=\; 0 \quad .$$

After matching infinities, the value of $c_0(k)$ follows:

$$\begin{aligned} c_0(k) &= -2ik \int_0^\infty dr\, r^2 \left\{ \varphi_1 - \frac{\Gamma(A_0^*)\, e^{\pi/ka}}{2}\; \varphi_2 \right\} \\ &= \frac{4\, ik\, \lambda\, \Gamma(A_0)}{a} \int_0^\infty dr\, (r \ln\, \mu r) \left\{ M^* U + \frac{\Gamma(A_0^*) e^{\pi/ka}}{2}\, |M|^2 \right\} . \end{aligned} \qquad (A.4)$$

The arguments in both Kummer functions $M,\ U$ are the same, e.g., $M = M(A_0, 2, 2ikr)$.

As we stated earlier, we are only interested in $R_{k0}(0)$. We then have,

$$R_{k0}(0) = e^{-\pi/2ka}\, \Gamma(A_0^*)\; [\; 1 + c_0(k)\;] \quad .$$

To complete the calculation it only remains to evaluate $c_0(k)$. We will come back to this later.

Repeating the above analysis for the case $\ell = 1$ gives:

$$\left.\frac{\overline{R}_{k1}}{r}\right|_{r=0} = [\; 1 + c_1(k)\;]\; \left.\frac{\widetilde{R}_{k1}}{r}\right|_{r=0} \quad , \qquad (A.5)$$

where

$$c_1(k) = \lambda\, \frac{(2ik)^3\; \Gamma(A_1)}{3\, a} \int_0^\infty dr\, r^3\; \ln{(\mu r)}\;\; M^* \left[U - \frac{e^{\pi/ka}\; \Gamma(A_1^*)}{12}\; M \right] \quad . \qquad (A.6)$$

In the above, $M, U \equiv M, U(A_1, 4, 2ikr)$, and A_1 was defined in Eq. (A.3). We now explain in some detail how $c_0(k)$ and $c_1(k)$ are evaluated.

1. Evaluation of $c_0(k)$.

Using the identity:

$$M(a, c, 2ikr) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{i\pi a} U(a, c, 2ikr) + \frac{\Gamma(c)}{\Gamma(a)} e^{i\pi(a-c)} e^{2ikr} U(c-a, c, -2ikr) \;, \qquad (A.6)$$

and making a change of variable: $r = \rho/2k$, we rewrite Eq. (A.4) as (below $\eta \equiv ka$):

$$
\begin{aligned}
c_0(k) =&\ \lambda\, \frac{-i\, e^{-\pi/\eta}}{2\,\eta\, |\,\Gamma(A_0)|^2}\, J_0 \quad , \\
J_0 \equiv& \int_0^\infty dr\; r\, \ln\left(\frac{r\mu}{2k}\right) \left[e^{-ir} U(A_0, 2, i\, r)^2\, \Gamma(A_0)^2 - \text{c.c.}\right] \quad ;
\end{aligned}
\tag{A.7}
$$

(c.c. $\equiv$ complex conjugate). If we now also rewrite $U^2\,\Gamma(A_0)^2$ as

$$
U(A_0, 2, ir)^2\, \Gamma(A_0)^2 = \left[U(A_0, 2, ir)^2\, \Gamma(A_0)^2 + \frac{1}{r^2}\right] \; - \; \frac{1}{r^2} \quad ,
$$

then the square bracket can be rotated to $r = -i\rho$ ($\rho > 0$) while its complex conjugate is rotated to $r = +i\rho$. Eq. (A.7) becomes:

$$
\begin{aligned}
J_0 = 2i& \int_0^\infty dr\; r\, \ln\frac{r\mu}{2k}\; \frac{\sin r}{r^2} \\
&- \int_0^\infty dr\; r\, e^{-r}\, \ln\frac{r\mu}{2k}\, \left[U(A_0, 2, r)^2\, \Gamma(A_0)^2 - \text{c.c.}\right] \\
&+ \frac{i\pi}{2} \int_0^\infty dr\; r\, e^{-r}\, \left[U(A_0, 2, r)^2\, \Gamma(A_0)^2 + U(A_0^*, 2, r)^2\, \Gamma(A_0^*)^2 - \frac{2}{r^2}\right] \; .
\end{aligned}
\tag{A.8}
$$

Note that each integral is convergent. If we define

$$
K_0(\epsilon) \equiv \int_0^\infty dr\; r^{1+\epsilon}\, e^{-r}\, U(A_0, 2, r)^2\, \Gamma(A_0)^2 \quad ,
\tag{A.9}
$$

(convergent for $\epsilon > 0$), we have:

$$
\begin{aligned}
J_0 = i\pi& \left(\ln\frac{\mu}{2k} - \gamma_E\right) \; - \; \frac{\partial}{\partial\epsilon}\left[K_0(\epsilon) - K_0(\epsilon)^*\right]_{\epsilon=0^+} \\
&- \ln\frac{\mu}{2k}\, \left[K_0(\epsilon) - K_0(\epsilon)^*\right]_{\epsilon=0^+} + \frac{i\pi}{2}\left[K_0(\epsilon) + K_0(\epsilon)^* - 2\,\Gamma(\epsilon)\right]_{\epsilon=0^+} \; .
\end{aligned}
\tag{A.10}
$$

To evaluate $K_0(\epsilon)$, we replace one of the U's by its integral representation

$$
e^{-r}\, U(a, c, r)\, \Gamma(a) = \int_0^1 dt\; e^{-r/t}\, t^{-c}\, (1-t)^{a-1} \quad ,
\tag{A.11}
$$

and use the result

$$
\begin{aligned}
\int_0^\infty dr\; r^{b-1}\, e^{-sr} U(a, c, r) = \frac{\Gamma(b-c+1)}{\Gamma(a)}\; s^{-b} \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n) \quad (1-s^{-1})^n,}{\Gamma(a+b-c+1+n)\Gamma(1+n)} \; , \\
\text{Re}(s) > 1/2 \; .
\end{aligned}
\tag{A.12}
$$

We find:

$$K_0(\epsilon) = \Gamma(1+\epsilon)^2 \sum_{n=0}^{\infty} \frac{\Gamma(2+n+\epsilon)\,\Gamma(A_0+n)^2}{\Gamma(1+n)\,\Gamma(A_0+n+1+\epsilon)^2} \quad , \tag{A.13}$$

which we rewrite for convenience as

$$K_0(\epsilon) = \Gamma(1+\epsilon)^2 \left\{ \zeta(1+\epsilon) + \sum_{n=0}^{\infty} \left[\frac{\Gamma(2+n+\epsilon)\,\Gamma(A_0+n)^2}{\Gamma(1+n)\,\Gamma(A_0+n+1+\epsilon)} - \frac{1}{(n+1)^{1+\epsilon}} \right] \right\} \tag{A.14}$$

We would like to point out that the remaining sum is convergent for $\epsilon = 0$, and its derivative with respect to ϵ also converges for $\epsilon = 0$. We then obtain the following partial results:

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon} \left[K_0(\epsilon) - K_0(\epsilon)^*\right]_{\epsilon=0} = \sum_{n=0}^{\infty} \left[\frac{1+n}{(A_0+n)^2} \Big(\psi(n+2) - 2\gamma_E - 2\psi(A_0+n+1)\Big) - \text{c.c.} \right], \\
&\left[K_0(\epsilon) - K_0(\epsilon)^*\right]_{\epsilon=0} = \sum_{n=0}^{\infty} \left[\frac{1+n}{(A_0+n)^2} - \frac{1+n}{(A_0^*+n)^2} \right] \quad , \\
&\left[K_0(\epsilon) + K_0(\epsilon)^* - 2\,\Gamma(\epsilon)\right]_{\epsilon=0} = \sum_{n=0}^{\infty} \left[\frac{1+n}{(A_0+n)^2} + \frac{1+n}{(A_0^*+n)^2} - \frac{2}{n+1} \right].
\end{aligned} \tag{A.15}$$

Putting everything together, we find:

$$\begin{aligned}
J_0 = &-\frac{i\pi}{2} \Big(2\,\gamma_E + \psi(A_0) + \psi(A_0^*) + (A_0-1)\,\psi'(A_0) + (A_0^*-1)\,\psi'(A_0^*) \Big) \\
&+ \left(\ln\frac{\mu}{2k} - 2\,\gamma_E \right) \Big(i\pi + \psi(A_0) - \psi(A_0^*) + (A_0-1)\,\psi'(A_0) - (A_0^*-1)\,\psi'(A_0^*) \Big) \\
&+ \sum_{n=0}^{\infty} \left[\frac{1+n}{(A_0+n)^2} \Big(2\psi(A_0+n+1) - \psi(n+2) \Big) - \text{c.c.} \right] \quad .
\end{aligned} \tag{A.16}$$

For small velocities (*i.e.* for $\eta = ka \lesssim 0.1$), we can use the asymptotic behavior of the ψ function and its derivatives to rewrite Eq. (A.16) as

$$J_0 = 2i\pi \left(\ln\frac{\mu a}{2} - 1 - 2\,\gamma_E + \frac{\eta^2}{12} + \frac{\eta^4}{40} + \ldots \right) \quad , \tag{A.17}$$

and our result for $c_0(k)$ is:

$$c_0(k) = \frac{\lambda}{2} \left[\ln\frac{\mu\,a}{2} - 1 - 2\,\gamma_E + \frac{\eta^2}{12} + \frac{\eta^4}{40} + \ldots \right] \quad . \tag{A.18}$$

For $\eta > 0.1$, the sum in Eq. (A.16) is evaluated numerically. See Fig. 1 for a plot of $c_0(k)$ as a function of η, when $\mu = 2/a$.

2. Evaluation of $c_1(k)$

For the case $\ell = 1$, we need to evaluate

$$c_1(k) = \lambda \frac{-i\, e^{-\pi/\eta}}{2\,\eta |\,\Gamma(A_1)|^2}\, J_1 \quad ,$$
$$J_1 \equiv \int_0^\infty dr\; r^3 \ln\left(\frac{r\mu}{2k}\right) \left[e^{-ir} U(A_1, 4, ir)^2\, \Gamma(A_1)^2 - \text{c.c.}\right] \quad . \tag{A.19}$$

J_1 is evaluated in the same maner as J_0. We first rewrite $U^2 \Gamma(A_1)^2$ as

$$U(A_1, 4, i\,r)^2\, \Gamma(A_1)^2 = \left[U(A_1, 4, i\,r)^2\, \Gamma(A_1)^2 - \mathcal{D}(ir)\right] + \mathcal{D}(ir) \quad ,$$
$$\mathcal{D}(z) \equiv \frac{4}{z^6} + \frac{12 - 4\,A_1^2}{z^5} + \frac{21 - 16\,A_1 + 3\,A_1^2}{z^4} \quad . \tag{A.20}$$

Then the square bracket is rotated to $r = -i\rho$, and its complex conjugate to $r = i\rho$. Eq. (A.19) becomes:

$$\begin{aligned} J_1 = &\int_0^\infty dr\; r\, \ln \frac{r\mu}{2k}\, \left[e^{-ir}\, \mathcal{D}(ir) - \text{c.c.}\right] \\ &+ \int_0^\infty dr\; r\, e^{-r}\, \ln \frac{r\mu}{2k}\, \left[U(A_1, 4, r)^2\, \Gamma(A_1)^2 - \mathcal{D}(r) - \text{c.c.}\right] \\ &- \frac{i\pi}{2} \int_0^\infty dr\; r\, e^{-r}\, \left[U(A_1, 4, r)^2\, \Gamma(A_1)^2 - \mathcal{D}(r) + \text{c.c.}\right] \quad , \end{aligned} \tag{A.21}$$

where each integral is convergent. The first integral is easily done and gives:

$$\mathcal{R} \equiv i \left(\ln \frac{\mu}{2k} - \gamma_E\right) \left(\frac{3\pi}{\eta^2} + \frac{8}{\eta} + \pi\right) + i \left(\frac{8}{\eta} + \pi\right) \quad . \tag{A.22}$$

To compute the remaining integrals, we define

$$K_1(\epsilon) \equiv \int_0^\infty dr\; r^{3+\epsilon}\, e^{-r}\, U(A_1, 4, r)^2\, \Gamma(A_1)^2 \quad ,$$
$$L_1(\epsilon) \equiv \int_0^\infty dr\; r^{3+\epsilon}\, e^{-r}\, \mathcal{D}(r) \quad , \tag{A.23}$$

which are convergent for $\epsilon > 2$. Eq. (16) is then rewritten as:

$$\begin{aligned} J_1 = \mathcal{R} + \lim_{\epsilon \to 0} \Big[&\Big(\frac{\partial}{\partial \epsilon} + \ln \frac{\mu}{2k} \Big) \Big(K_1(\epsilon) - K_1(\epsilon)^* - L_1(\epsilon) + L_1(\epsilon)^* \Big) \\ &- \frac{i\pi}{2} \Big(K_1(\epsilon) + K_1(\epsilon)^* - L_1(\epsilon) - L_1(\epsilon)^* \Big) \Big] \quad . \end{aligned} \tag{A.24}$$

Note that the result for $\epsilon = 0$ is obtained by analytical continuation. The results for K_1 and L_1 are easily obtained if one uses Eqs. (A.11) and (A.12):

$$\begin{aligned} L_1(\epsilon) &= 4\,\Gamma(-2+\epsilon) + (12 - 4\,A_1)\,\Gamma(-1+\epsilon) + (21 - 16\,A_1 + 3\,A_1^2)\,\Gamma(\epsilon) \quad , \\ K_1(\epsilon) &= \Gamma(1+\epsilon)^2 \sum_{n=0}^{\infty} \frac{\Gamma(4+n+\epsilon)\,\Gamma(A_1+n)^2}{\Gamma(1+n)\,\Gamma(A_1+n+1+\epsilon)^2} \quad . \end{aligned} \tag{A.25}$$

To extract the pole part of $K_1(\epsilon)$, we rewrite the last equation as:

$$K_1(\epsilon) = \Gamma(1+\epsilon)^2 \sum_{n=0}^{\infty} u_n(\epsilon) + \Gamma(1+\epsilon)^2 \sum_{n=0}^{\infty} \left[\frac{\Gamma(4+n+\epsilon)\,\Gamma(A_1+n)^2}{\Gamma(1+n)\,\Gamma(A_1+n+1+\epsilon)^2} - u_n(\epsilon) \right] \ , \tag{A.26}$$

where

$$\begin{aligned} u_n(\epsilon) \equiv & \frac{1}{(n+1)^{-1+\epsilon}} + \frac{5 - 2\,A_1 + \epsilon(7/2 - 2\,A_1)}{(n+1)^{\epsilon}} \\ & + \frac{11 - 12A_1 + 3A_1^2 + \epsilon(175/12 - 18A_1 + 5A_1^2)}{(n+1)^{1+\epsilon}} \quad . \end{aligned} \tag{A.27}$$

The first sum which contains the pole part gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(\epsilon) = & \ \zeta(-1+\epsilon) + \zeta(\epsilon) \ \left[5 - 2\,A_1 + \epsilon\left(\frac{7}{2} - 2\,A_1\right)\right] \\ & + \zeta(1+\epsilon)\left[11 - 12\,A_1 + 3\,A_1^2 + \epsilon\left(\frac{175}{12} - 18\,A_1 + 5\,A_1^2\right)\right] \quad . \end{aligned} \tag{A.28}$$

In the above equation, ζ is the Riemann's zeta function. Now the second sum in Eq. (A.26) converges for $\epsilon = 0$ and so does its derivative with respect to ϵ. We obtain the following partial results:

$$\begin{aligned} & \sum_{n=0}^{\infty} \left[\frac{\Gamma(4+n+\epsilon)\,\Gamma(A_1+n)^2}{\Gamma(1+n)\,\Gamma(A_1+n+1+\epsilon)^2} - u_n(\epsilon) \right] \Bigg|_{\epsilon\to 0} \\ & \quad = \ i\,\left(\frac{1}{\eta^3} + \frac{1}{\eta}\right)\ \psi'(A_1) + \left(1 + \frac{3}{\eta^2}\right)\ \Big(\psi(A_1) + \gamma_E\Big) \quad , \end{aligned} \tag{A.29}$$

$$\begin{aligned} & \frac{\partial}{\partial\epsilon} \sum_{n=0}^{\infty} \left[\frac{\Gamma(4+n+\epsilon)\,\Gamma(A_1+n)^2}{\Gamma(1+n)\,\Gamma(A_1+n+1+\epsilon)^2} - u_n(\epsilon) - \text{c.c.} \right] \Bigg|_{\epsilon\to 0} = \\ & \left(2 + \frac{5}{\eta^2}\right)\ \Big(\psi(A_1) - \psi(A_1^*)\Big) - \frac{9\,i}{\eta}\ \Big(\psi(A_1) + \psi(A_1^*)\Big) - \frac{2\,i}{\eta}\Big(\ln 2\pi - \frac{21}{2} + 8\,\gamma_E\Big) \\ & + i\,\left(\frac{1}{\eta^3} + \frac{1}{\eta}\right)\ \sum_{n=0}^{\infty} \left[\frac{\psi(n+4) - 2\psi(A_1+n+1)}{(A_1+n)^2} + \text{c.c.}\right] \\ & - \left(1 + \frac{3}{\eta^2}\right)\ \sum_{n=0}^{\infty} \left[\frac{\psi(n+4) - 2\psi(A_1+n+1)}{A_1+n} - \text{c.c.}\right] \quad , \end{aligned} \tag{A.30}$$

where we have used $A_1 = 2 + i/\eta$. For small velocities, we have:

$$J_1 = 2\,i\,\pi \left[\left(\ln \frac{\mu a}{2} - 2\,\gamma_E \right) \left(1 + \frac{3}{\eta^2} \right) + \frac{4}{\eta^2} - \frac{1}{12} + \frac{11}{120}\,\eta^2 - \frac{1}{\eta\,\pi} \right] + \mathcal{O}(\eta^4)\ , \qquad \text{(A.31)}$$

and our result for $c_1(k)$, $\eta \lesssim 0.1$, is:

$$\begin{aligned} c_1(k) = \lambda \Bigg[& \left(\ln \frac{\mu a}{2} - 2\,\gamma_E \right) \left(-\frac{3}{2} + \eta^2 - \eta^4 \right) - 2 + \frac{49}{24}\,\eta^2 - \frac{167}{80}\,\eta^4 \\ & + \frac{1}{2\,\pi} \left(\eta - \eta^3 + \eta^5 \right) + \mathcal{O}(\eta^6) \Bigg] \ . \end{aligned} \qquad \text{(A.32)}$$

See Fig. 2 for a plot of $c_1(k)$ as a function of η, when $\mu = 2/a$.

Before we close this section, we would like to add a few words about the correctness of the results presented here. We have checked that the terms which are proportional to $\ln \mu$ in Eqs. (A.19, A.20) are the same as those obtained by solving Eq. (A.1) directly, with the obvious replacement $\ln \mu r \to \ln \mu$; they can be deduced from Eq. (A.2), with $r = 0$, and $a^{-1} \to a^{-1} + a^{-1} \lambda \ln \mu$. These terms are needed to show that the physical results (see for instance Eq. (3.9)) are, to the order we are working, μ-independent (this is of course a consequence of the μ-independence of the Hamiltonian H in Eq. (3.2)). We have also checked that Eqs. (A.18, A.32) reproduce the correct results in the limit $k \to 0$. The latter are obtained as follows: one first regulates the integrals in Eqs. (A.7, A.19) (e.g., by adding a term $e^{-r\epsilon}$ in the integrands), then one uses

$$\lim_{a \to \infty} \Gamma(a - c + 1)\; U(a,\ c,\ z/a) = 2\; z^{1/2 - c/2}\; K_{c-1} \left(2\, z^{1/2} \right) \quad ,$$

where K_{c-1} is the Bessel function of the second kind. The integrals one obtains involve a product of Bessel functions, and are easily done. This provides a crucial test of the correctness of the calculation.

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FIGURES

FIG. 1.

c_0 versus $\eta \equiv ka$.

FIG. 2.

c_1 versus $\eta \equiv ka$.

FIG. 3.

R_c^V v.s. $\sqrt{s}$, with $m_c = 1.57\ GeV$, $\Lambda_{QCD} = 0.2\ GeV$, $\langle \alpha_s\, G^2 \rangle = 0.042\ GeV^4$. The solid and dotted lines were obtained with the choice $\mu = 2\, m_c$; the dashed-line was obtained by choosing μ such that the radiative corrections vanish (*i.e.* such that $c_0 = 0$). This shows that R_c^V is not very sensitive to the choice of μ.

FIG. 4.

Theoretical predictions for R_c^V together with experiment results (Ref. [15], Particle Data Book). The dotted-line represents the parton model prediction, and is obtained with $m_c = 1.57\ GeV$, $\mu = 2\,m_c$, $\Lambda_{QCD} = 0.2\ GeV$, and $\langle \alpha_s\, G^2 \rangle = 0.042\ GeV^4$. The solid line represents the "exact" value of R_c^V and the gray area surrounding it represents the uncertainties which are obtained by varying m_c, Λ_{QCD} and $\langle \alpha_s\, G^2 \rangle$. The dashed-line gives R_c when radiative and nonperturbative corrections are neglected. The experimental data is represented by the diamonds; the horizontal error is less than 4% while the vertical errors are large ($\sim 5 - 35\%$).

FIG. 5.

R_b^V v.s. $\sqrt{s}$. Same conventions as in Fig. 4.

FIG. 6.

Prediction for R_t^V v.s. $\sqrt{s}$, with $\mu = M_Z$, $\alpha_s = 0.119$, and $\langle \alpha_s G^2 \rangle = 0.042\ GeV^4$. $R_t^{V(exact)}$ is plotted for $m_t = 165\ GeV$ (solid line), $m_t = 180\ GeV$ and $m_t = 150\ GeV$ (dashed lines).

FIG. 7.

Prediction for R_t^A v.s. $\sqrt{s}$, with $m_t = 165\ GeV$, and $\langle \alpha_s G^2 \rangle = 0.042\ GeV^4$. We plot $R_t^{V(exact)}$ using two different criterias for the choice of μ: (a) choose $\mu = M_Z$ and $\alpha_s = 0.119$ (solid line); (b) choose μ such that the radiative corrections vanish, *i.e.* such that $c_1 = 0$ (dashed line).